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# On the Irreducible Components of the Singular Locus of $A_g$

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## 0. INTRODUCTION

Let $A_g$ be the moduli space of principally polarized abelian varieties (p.p.a.v. for short) of dimension $g$ ($g \geq 3$) over an algebraically closed field $k$. The aim of this paper is to describe the irreducible components of the singular locus Sing $A_g$ of $A_g$. It is well known that this singular locus

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consists of those abelian varieties with automorphism group different from $\{\pm 1\}$ (see [8]). The analogous problem for the moduli space of algebraic curves was studied in [3].

Let $(X, \Theta)$ be a p.p.a.v.; if the finite group $G = Aut(X, \Theta)$ is non-trivial ($\neq \{\pm 1\}$), and $p$ is a prime divisor of the order of $G$, then $G$ contains an element of order $p$. Hence, the study of the irreducible components of Sing $A_g$ can be reduced to the study of the set of those abelian varieties admitting an automorphism of prime order $p$.

The general plan of the paper is as follows: in Section 1 we develop the deformation theory (local and global) of a triple $(X, \Theta, \rho)$ consisting of a p.p.a.v. $(X, \Theta)$ and $\rho \in Aut(X, \Theta)$, $\rho^p = id$. We can associate to $(X, \Theta, \rho)$ several discrete invariants. First of all the differential map

$$d\rho : T_0X \to T_0X$$

can be diagonalized if $p \neq \operatorname{char} k$. Thus, in a suitable base of $T_0X$ we can write $d\rho = diag(\xi^{k_1}, \ldots, \xi^{k_g})$, $\xi$ a primitive root of 1, and $0 \leq k_i < p$. Moreover, we have a decomposition

$$T_0X = V_0 \oplus V_1 \oplus \cdots \oplus V_{p-1},$$

where $V_i$ is the subspace in which $\rho$ acts by multiplication by $\xi^i$. Call $n_i = \dim V_i$ (note that $n_i$ can be zero for some $i$). Then the local deformation space of $(X, \Theta, \rho)$ can be explicitly described in terms of the discrete invariants $(k_1, \ldots, k_g)$ and $(n_0, \ldots, n_{p-1})$ associated to $d\rho$. This description is contained in Theorem 1.2. The general technique used in the proof of this theorem is based on the classical papers [7, 8].

Moreover, if $n \geq 3$, the restriction of $\rho$ to the $n$-torsion points $X_n$ of $X$ gives an automorphism

$$\rho_n : X_n \xrightarrow{\sim} X_n;$$

note that $\rho_n \in GL(2g, \mathbb{Z}/n\mathbb{Z})$. The conjugation class of $\rho_n$ determines the global deformation space of $(X, \Theta, \rho)$. If we denote by $A_g(p, \rho_n)$ the set of all p.p.a.v. admitting an automorphism of order $p$ which acts on $X_n$ with class $\rho_n$, then it is a consequence of the deformation theory (in the case $k = \mathbb{C}$) that the family of subsets

$$A_g(p, \rho_n) \subset A_g$$

is a family of irreducible algebraic subvarieties that cover the singular locus Sing $A_g$. The dimension of these subvarieties can be computed with the aid of the local deformation theory.

The second section of the paper is devoted to the study of the possible inclusions of the type

$$A_g(p, \rho_n) \subset A_g(q, \sigma_n).$$

There are several examples of this kind of inclusions (Subsection 2.1). Subsection 2.2 is strongly based on [1], where the extreme case $g = (p-1)/2$ is studied in detail. We concentrate our study on the case $n_0 = 0$ and $k = \mathbb{C}$. The key point is Lemma 2.7, which states that any $A_g(p, \rho_n)$ contains some product

$$Z_h = X_1^h \times \cdots \times X_r^h,$$

of p.p.a.v. of dimension $(p-1)/2$ with $\mathbb{Z}_p$-action.

This lemma is essential in two directions. First of all, it gives an answer to the problem of which automorphism of order $p$ can appear like an automorphism of some $(X, \Theta)$. The answer is that the only restrictions are the elementary numerical relations established in Proposition 2.1 (in the case $n_0 = 0$). On the other hand, the lemma allow us to construct some kind of inductive argument in order to reduce our research to the extreme case studied in [1].

The general idea is to establish the injection

$$Aut(X, \Theta) \hookrightarrow Aut(Z_h, \Theta_h).$$

This injection is proved in Lemma 2.10.

These inclusions restrict the groups that can appear like the automorphism group of a general element of $A_g(p, \rho_n)$. Using this technique we describe, in Theorem 2.14, a class of varieties $A_g(p, \rho_n)$ such that its general element $(X, \Theta)$ satisfies $Aut(X, \Theta)/\{\pm 1\} = \mathbb{Z}_p$.

Finally we would like to emphasize the main conclusions of the paper:

(1) The local deformation theory of $(X, \Theta, \rho)$ is determined by the analytical representation $d\rho$ (Theorem 1.2); this is valid on any algebraically closed field of characteristic $\neq p$.

(2) The global deformation theory of $(X, \Theta, \rho)$ is determined by the $n$-torsion representation $\rho_n(k = \mathbb{C})$ (Proposition 1.3 and Theorem 1.5).

(3) In general, it seems to be a difficult problem to determine $Aut(X, \Theta)$ for $(X, \Theta)$ a general element of $A_g(p, \rho_n)$. However, if we assume $n_0 = 0$ and $k = \mathbb{C}$, then:

(a) the problem can be reduced to some variations of the extreme case $g' = (p-1)/2$,

(b) using this reduction argument the problem can be solved in some particular cases (Theorem 2.14).

## 1. DEFORMATION THEORY OF p.p.a.v. WITH A CYCLIC AUTOMORPHISM OF ORDER $p$

1.1. *Local Theory.* Let $(X_0, \Theta_0)$ be a p.p.a.v., $\lambda_{\Theta_0}$ the associated isomorphism

$$\lambda_{\Theta_0} : X_0 \xrightarrow{\sim} X_0^t.$$

Assume there exists $\rho_0 : X_0 \to X_0$, an automorphism of p.p.a.v. of prime order $p$.

The local deformation functor of $(X_0, \lambda_{\Theta_0}, \rho_0)$ is defined as

$$P_0 : \mathscr{C} \to Sets,$$

$$P_0(R) = \left\{ \begin{array}{c} \text{equivalence classes} \\ (X, \varphi, \Theta, \rho) \end{array} \middle| \begin{array}{c} X \text{ abelian scheme}/R, \varphi : X \otimes k \xrightarrow{\sim} X_0, \\ \lambda_\Theta \otimes k = \lambda_{\Theta_0},\ \rho \in Aut(X) \\ \text{s.t. } \rho \otimes k \in \langle \rho_0 \rangle \end{array} \right\},$$

where $\mathscr{C}$ is the category of local artinian algebras. The precise meaning of the word "equivalence" in the definition is $(X, \varphi, \Theta, \rho) \sim (Y, \phi, \Theta', \theta)$, if and only if there exists

$$\begin{tikzcd} \Phi : X \arrow[r, "\sim"] \arrow[dr] & Y \arrow[d] \\ & \operatorname{Spec} R, \end{tikzcd}$$

such that $\Phi \otimes k = id$,

$$\begin{tikzcd} X \arrow[r, "\Phi"] \arrow[d, "\lambda_\Theta"] & Y \arrow[d, "\lambda_{\Theta'}"] \\ X^t & Y^t, \arrow[l, "\Phi*"] \end{tikzcd}$$

commutes, and $\Phi^{-1} \circ \theta \circ \Phi \in \langle \rho \rangle$.

The main tool for our study of $P_{\rho_0}$ is the rigidity lemma [6, Lemma 6.2]. As a consequence of this lemma we have the following:

*Remarks* 1.1. (1) If $(X, \varphi, \Theta, \rho) \in P_{\rho_0} R$, then

$$\begin{tikzcd} X \arrow[r, "\rho"] \arrow[d, "\lambda_\Theta"] & X \arrow[d, "\lambda_{\Theta'}"] \\ X^t \arrow[r, "\rho*"] & X^t, \end{tikzcd}$$

is commutative and $\rho^p = id_X$.

(2) If

$$P(R) = \left\{ \begin{array}{c|c} \text{equivalence classes} & X \text{ abelian scheme}/R, \varphi : X \otimes k \xrightarrow{\sim} X_0, \\ (X, \varphi, \Theta) & \lambda_\Theta \otimes k = \lambda_{\Theta_0}, \end{array} \right\},$$

then $P_{\rho_0}$ is a subfunctor of $P$. Indeed,

$$k \otimes (\Phi^{-1} \circ \theta \circ \Phi) \in \langle \rho_0 \rangle = \langle \rho \otimes k \rangle.$$

Thus, using the rigidity lemma, we get $(\Phi^{-1} \circ \theta \circ \Phi) \in \langle \rho \rangle$.

Note that the map

$$d\rho_0 : T_0 X_0 \to T_0 X_0$$

satisfies $(d\rho_0)^p = id$. Thus it can be represented (if $p \neq char\, k$) in suitable coordinates like a diagonal matrix, $A = diag(\xi^{k_i})_{i=1,\dots g}$, where $\xi$ is a primitive $p$-root of 1 and $0 \leq k_i \leq p-1$.

Our goal is to prove the following:

THEOREM 1.2. *The functor $P_{\rho_0}$ is pro-representable and formally smooth; it is pro-represented by $k[[t_{ij}]]_{i \leq j}/\mathfrak{a}$, where $\mathfrak{a}$ is the ideal generated by $\langle t_{ij} \mid k_j \neq p - k_i \rangle$.*

*Proof.* In [7, 8], it was proved that $P$ (defined in the Remark) is pro-represented by the complete $k$-algebra $k[[t_{11}, \dots, t_{gg}]]/\langle t_{ij} - t_{ji} \rangle_{i<j}$. Call $\mathcal{O}$ this algebra; then, if $P_0$ is pro-representable, it must be pro-represented by $\mathcal{O}_{/}\mathfrak{a}$, $\mathfrak{a}$ being some ideal.

Assume that $\pi : R \to R' \to 0$ is small, that is, $R, R' \in \mathcal{C}$ and $I = \operatorname{Ker} \pi$ satisfies $I \cdot m_R = 0$. Note that, given any surjection $\pi$ in $\mathcal{C}$, it can be factorized through a finite number of small surjections.

Let $0 \to I \to R \to R' \to 0$ be a small surjection, and let $(X', \Theta', \rho') \in P_0(R')$. If $(X, \Theta) \in P(R)$, then $\rho'$ lifts to $(X, \Theta)$ if and only if

$$d\rho_0(\eta) = \rho_0^*(\eta),$$

where $d\rho_0$ and $\rho_0^*$ are the natural endomorphisms of $H^1(X_0, \mathcal{T}_{X_0}) \otimes_k I$ induced by $\rho_0$, and $\eta \in H^1(X_0, \mathcal{T}_{X_0}) \otimes I$ is the class of deformation corresponding to

$$\begin{array}{ccc} X & \longrightarrow & X' \\ \downarrow & & \downarrow \\ \operatorname{Spec} R & \longrightarrow & \operatorname{Spec} R', \end{array} \tag{1.1}$$

(see [8, p. 9]).

Now, we use the fact that $H^1(X_0, \mathcal{T}_{X_0}) \simeq T_0 X_0 \otimes T_0 X_0^*$.

Call $\bar{a}$ the base in which $d\rho_0$ is represented by $A = diag(\xi^{k_i})_{1=1,\dots g}$, and $\bar{b} := d\lambda_{\Theta_0}(\bar{a})$ the compatible base of $T_0X_0^*$. With these choices of coordinates we have $\rho_0^* = diag(\xi^{p-k_i})$.

In these coordinates the condition 1.1 is expressed as

$$E \cdot diag(\xi^{k_i}) = diag(\xi^{p-k_i}) \cdot E,$$

where $E = (\eta_{ij})$, $\eta_{ij} \in T_0X_0 \otimes T_0X_0^* \otimes I$ the coordinates corresponding to $\eta$.

Call $\mathfrak{a}$ the ideal in $\mathcal{O}$ generated by the relations

$$T \cdot diag(\xi^{k_i}) = diag(\xi^{p-k_i}) \cdot T,$$

$T = (t_{ij})$.

Then we conclude that if $\pi : R \to R' \to 0$ is small, $(X', \Theta', \rho') \in P_0(R')$, and

$$\mathbb{L}(\pi, X', \Theta', \rho') := \left\{ \begin{array}{c|c} \text{equivalence classes} & (X, \Theta, \rho) \to \operatorname{Spec} R \\ (X, \varphi, \lambda_\Theta, \rho) & \text{lifts to } (X', \Theta', \rho') \end{array} \right\};$$

then

$$\mathbb{L}(\pi, X', \Theta', \rho') \overset{1:1}{\to} \{\alpha \in \operatorname{Hom}_k(\mathcal{O}_{/}\mathfrak{a}, R) \mid \alpha(t_{ij}) \in I\}.$$

The previous bijection is given in the following natural way.

If $(X, \varphi, \Theta, \rho) \in \mathbb{L}(\pi, X', \Theta', \rho')$ define $\alpha$ by means of the assignation $\alpha(t_{ij}) = \eta_{ij}$. Conversely, if $\alpha(t_{ij}) := \eta_{ij} \in I$, the relations

$$E \cdot diag(\xi^{k_i}) = diag(\xi^{p-k_i}) \cdot E$$

$(E = (\eta_{ij}))$ are obviously verified, and thus there exist $\rho : X \to X$, lifting to $\rho'$. This $\rho$ must be an automorphism [7, Lemma 2.2.2], and the bijection is given.

Now, it is easy to deduce that $\mathcal{O}_{/}\mathfrak{a}$ pro-represent to $P_0$. Indeed, given any $R \in \mathscr{C}$ and $\pi : R \to k \to 0$ decompose $\pi$ in a chain $R \xrightarrow{\pi_n} R_n \to \cdots R_1 \xrightarrow{\pi_0} k \to 0$ of small surjections. Moreover, given $(X, \Theta, \rho) \in P_0(R)$ we have a natural chain of deformations,

$$\begin{array}{ccccc}
(X_0, \Theta_0, \rho_0) & \longrightarrow & (X_1, \Theta_1, \rho_1) & \cdots \longrightarrow & (X, \Theta, \rho) \\
\downarrow & & \downarrow & & \downarrow \\
\operatorname{Spec} k & \longrightarrow & \operatorname{Spec} R_1 & \cdots \longrightarrow & \operatorname{Spec} R,
\end{array}$$

obtained by base change.

Then $(X, \Theta, \rho) \in \mathbb{L}(\pi_n, X_n, \Theta_n, \rho_n)$, and using the previous bijection we obtain an element of $\operatorname{Hom}_k (\mathcal{O}/\mathfrak{a}, R)$.

Conversely, given $\alpha : \mathcal{O}/\mathfrak{a} \to R$, composition gives rise to a map $\bar{\alpha} : \mathcal{O}/\mathfrak{a} \to R_1$, and as $\bar{\alpha}(t_{ij}) \in m_{R_1} = \operatorname{Ker} \pi_0$ this morphism defines $(X_1, \Theta_1, \rho_1) \in P_0(R_1)$. Iterating this construction we obtain $(X, \Theta, \rho) \in P_0(R)$.

It is easy to check that these assignments define an isomorphism

$$P_0(-) \xrightarrow{\simeq} \operatorname{Hom}_k(\mathcal{O}/\mathfrak{a}, -). \qquad \blacksquare$$

If we decompose $T_0X = V_0 \oplus V_1 \oplus \cdots \oplus V_{p-1}$, where $V_i$ is the subspace in which $\rho$ acts by multiplication by $\xi^i$, and we call $n_i = \dim V_i$, then it follows from the theorem that if $p > 2$ then

$$\dim \mathcal{O}/\mathfrak{a} = \frac{n_0(n_0+1)}{2} + \sum_{i=1}^{p-1/2} n_i \cdot n_{p-i}.$$

1.2. *Global Deformation Theory.* Given $n \geq 3$, a prime number $p (p \nmid n)$, and an element

$$\rho_n \in GL(2g, \mathbb{Z}/n\mathbb{Z}),$$

define the functor

$$\mathcal{A}_{g,n}(p, \rho_n) : \mathit{Schem} \to \mathit{Sets},$$

that assigns to each $S$ the equivalence classes of $(X, \Theta, s_i, p, \rho_n)$, where:

(a) $X \to S$ is an abelian variety with principal polarization $\Theta$,

(b) $s_i : S \to X$ is a $n$-level structure (thus, we have an inclusion

$$\mathcal{A}_{g,n}(p, \rho_n) \subset \mathcal{A}_{g,n},$$

where $\mathcal{A}_{g,n}$ is the functor defined in [6, 7.2]),

(c) there exists a $\mathbb{Z}_p$-action on $X/S$ satisfying that the induced action on $(X_n)_s$ in the base determined by $\{s_i\}$ is given by $\langle \rho_n \rangle$ for any closed point $s \in S$.

PROPOSITION 1.3. *$\mathcal{A}_{g,n}(p, \rho_n)$ is represented by a smooth closed subscheme*

$$A_{g,n}(p, \rho_n) \subset A_{g,n}$$

*of dimension* $\dim A_{g,n}(p, \rho_n) = (n_0(n_0+1)/2) + \sum_{i=1}^{p-1/2} n_i \cdot n_{p-i}$.

*Proof.* We need to prove the existence of a subscheme $A_{g,n}(p, \rho) \subset A_{g,n}$ such that for any morphism

$$T \xrightarrow{f} A_{g,n},$$

$f$ factorizes through $A_{g,n}(p, \rho_n)$ if and only if the corresponding family $X \to T$ is an element of $\mathcal{A}_{g,n}(p, \rho_n)(T)$.

First of all, there exists a subset of $A_{g,n}$ whose closed points are in 1:1 correspondence with $k$-rational points of $A_{g,n}$ admitting an automorphism of order $p$ whose $X_n$-representation is an element of $\langle\rho_n\rangle$. But this set is readily identified with

$$\left\{(X, s_i) \in A_{g,n} \mid \rho_n^k \cdot (X, s_i) = (X, s_i), \quad k = 1, \ldots, p-1\right\};$$

thus, it is a closed subset of $A_{g,n}$. We denote this subscheme as $A_{g,n}(p, \rho_n)$, and using Theorem 1.2 we conclude that $A_{g,n}(p, \rho_n)$ is smooth, as it is a fine moduli scheme. ∎

*Remark* 1.4. The identification of the tangent space to $A_{g,n}$ at any point $[X]$ with $\mathfrak{m}/\mathfrak{m}^2$, with $\mathfrak{m}$ standing for the maximal ideal of $\mathcal{O}/\mathfrak{a}$, can be interpreted, in more geometrical terms, as the identification of this tangent space with $S^2(T_0X)^{\mathbb{Z}/(p)}$. To obtain this identification note that the tangent space to $A_{g,n}$ at the point $X$ is isomorphic to $S^2(T_0X)$.

The next step is to prove that $A_{g,n}(p, \rho_n)$ is a connected scheme. It will be proved in the case $k = \mathbb{C}$. In this case we have a commutative diagram,

$$\begin{tikzcd} \mathbb{H}_g \arrow[r, "v"] \arrow[dr, "\pi"'] & A_{g,n}(p, \rho_n) \arrow[d, "\pi_n"] \\ & A_g \end{tikzcd}$$

where $\mathbb{H}_g$ denotes the Siegel half-plane of dimension $g$.

Recall that given $(I, \tau)$, $\tau \in \mathbb{H}_g$, representing the matrix period of a principally polarized abelian variety $X_\tau$ over $\mathbb{C}$, if $X_\tau$ admits an automorphism $\sigma : X_\tau \to X_\tau$, then the analytic representation of $\sigma$,

$$A : T_0X_\tau \to T_0X_\tau,$$

and the rational representation

$$R : \Lambda \to \Lambda$$

($\Lambda$ the $\mathbb{Z}$-module generated by the columns of $(I, \tau)$) are related by the expression

$$A(I, \tau) = (I, \tau)R.$$

Define

$$\mathbb{H}_g(p, \rho_n) = \{\tau \in \mathbb{H}_g \mid A(1/n)(I, \tau) = 1/n(I, \tau)R_n\},$$

where $R_n \in GL(2g, \mathbb{Z}/n\mathbb{Z})$ is the matrix corresponding to $\rho_n$.

Then, $\mathbb{H}_g(p, \rho_n)$ is a connected set, since $\mathbb{H}_g$ is a linearly convex space. Now, the image of $\mathbb{H}_g(p, \rho_n)$ under $v$ is just $A_{g,n}(p, \rho_n)$. Thus, $A_{g,n}(p, \rho_n)$ is a connected set in the analytical topology. This implies that $A_{g,n}(p, \rho_n)$ is connected in the Zariski topology too.

From this and Proposition 1.3 we have obtained:

THEOREM 1.5. *If $k = \mathbb{C}$, then $A_g(p, \rho_n)$ is an irreducible algebraic variety.*

Note, moreover, that $A_g(p, \rho_n) = A_g(p, \rho'_n)$ if and only if $\rho_n$ is conjugate to $\rho'_n$ in $GL(2g, \mathbb{Z}/n\mathbb{Z})$.

In order to see that all the irreducible components of the singular locus of $A_g$ arise among the $A_g(p, \rho_n)$, we need the following:

LEMMA 1.6. *Let $S$ be a connected scheme and $X$ an abelian scheme over $S$. If $X/S$ admits an action of $G$, then*

(a) *the analytical representation $A_s$ of the group on the tangent space to $X_s$ is independent of $s$, where $s$ varies on the set of closed points of $S$,*

(b) *the $(X_s)_n$-representation of the group $G$ on $GL(2g, \mathbb{Z}/n\mathbb{Z})$ is independent of $s$, where $s$ varies on the set of closed points of $S$.*

*Proof.* (a) Let us denote by $e: S \to X$ the section defining the neutral element; one has a canonical isomorphism

$$\pi_*\Omega_{X/S} \simeq e^*\Omega_{X/S},$$

$\Omega_{X/S}$ being the sheaf of relative differentials, with respect to $\pi: X \to S$. Then $\pi_*\Omega_{X/S}$ is a locally free sheaf over $S$ of rank $g$. Let us denote by $V(\mathscr{E}_{\mathscr{O}_S}(\pi_*\Omega_{X/S}))$ the vector bundle defined by the locally free sheaf $\mathscr{E}_{\mathscr{O}_S}(\pi_*\Omega_{X/S})$ and let $Aut_S(\pi_*\Omega_{X/S})$ be the subscheme of $V(\mathscr{E}_{\mathscr{O}_S}(\pi_*\Omega_{X/S}))$ representing the automorphisms of $\pi_*\Omega_{X/S}$. Let us denote by $\underline{G}$ the finite group scheme over $\mathbb{C}$ defined by $G$. Given a noetherian connected $\mathbb{C}$-scheme $S$ one has a natural morphism of group schemes over $S$:

$$\begin{tikzcd} \underline{G}_S \arrow[r, "\phi"] \arrow[dr] & Aut_S(\pi_*\Omega_{X/S}) \arrow[d] \\ & S \end{tikzcd}$$

There exists a finite covering $\{U_i\}$ by connected affine schemes such that $\pi_*\Omega_{X/S}|_{U_i} \simeq V_i \otimes_{\mathbb{C}} \mathscr{O}_{U_i}$, $V_i$ being a $g$-dimensional space; for each $V_i$ one has

$$Aut_S(\pi_*\Omega_{X/S})_{U_i} = \underline{GL}(V_i) \times U_i \subset \mathbb{A}^{g^2}_{U_i},$$

where $\underline{GL}(V_i)$ is the group $\mathbb{C}$-scheme defined by the linear group $GL(V_i)$.

For any $i$, $\rho$ induces a morphism of group schemes:

$$\underline{G}_{U_i} \overset{\rho_i}{\to} \underline{GL}(V_i) \times U_i.$$

To prove that $\rho_i(s)$ is independent of $s$ we can assume that $G$ is a cyclic group of order $n$. Let $G^*$ be the Cartier dual of $G$, the span of representations of dimension $g$ of $G$ is the $\mathbb{C}$-scheme:

$$R_g(\underline{G}) = \bigcup_{m_1+\cdots+m_s=g} \underline{G}^{*m_1} \times \cdots \times \underline{G}^{*m_s}.$$

The morphism $\rho_i$ induces a morphism

$$U_i \xrightarrow{\epsilon_i} R_g(\underline{G}),$$
$$s \to \epsilon(\rho_i(s)).$$

From the connectivity of $U_i$ we deduce that $\epsilon_s$ is constant for every $s$.

(b) The $G$-action on $(X_s)_n$ induces a morphism

$$S \xrightarrow{\rho} \underline{GL}(2g, \mathbb{Z}/n\mathbb{Z}) \times S,$$
$$s \to ((\rho_s)_n, s).$$

The decomposition of $\underline{GL}(2g, \mathbb{Z}/n\mathbb{Z}) \times S$ in connected components is

$$\bigcup_{g \in GL(2g, \mathbb{Z}/n\mathbb{Z})} \{g\} \times S.$$

The assertion follows at once from the continuity of $\rho$. ∎

## 2. IRREDUCIBLE COMPONENTS OF THE SINGULAR LOCUS OF $A_g$

2.1. *Examples of Inclusions.* In the sequel we assume $k = \mathbb{C}$. Let $(X, \Theta, \rho)$ be a point of $A_{g,n}(p, \rho_n)$ and let $V = T_0X$. The representation of $d\rho$ in $V$ is given by

$$V = V_0 \oplus V_1 \oplus \cdots \oplus V_{p-1},$$

$V_i$ being the subspace where $\rho$ acts by multiplication by $\xi^i$ ($\xi$ being a $p$-primitive root of 1). Let us denote $n_i = \dim V_i$.

Let $E$ be the integral representation of $\rho$ in the lattice $U$ defining the abelian variety like a complex torus.

Let $p(x)$ be the characteristic polynomial of $\rho$ in $V$ and $R(x)$ the characteristic polynomial of $\rho$ in $E$. One has

$$p(x) = \prod_{i=0}^{p-1} (x - \xi^i)^{n_i},$$

$$R(x) = p(x)\bar{p}(x) = \prod_{i=0}^{p-1} (x - \xi^i)^{n_i} \prod_{i=0}^{p-1} (x - \xi^{p-i})^{n_i} = \prod_{i=0}^{p-1} (x - \xi^i)^{n_i + n_{p-i}}$$

(where $n_p = n_0$).

That is, one has

$$R(x) = (x-1)^{2n_0} \cdot \prod_{i=1}^{p-1} (x - \xi^i)^{n_i + n_{p-i}}.$$

PROPOSITION 2.1. *There exists a natural number r such that $n_i + n_{p-i} = r$, for all $i \neq 0$ and $g = n_0 + ((p-1)/2)r$.*

*Proof.* Consider the decomposition of the $\langle \rho \rangle$-module $E$,

$$E = E_0 \oplus \overline{E},$$

where $E_0$ is the $\mathbb{Z}$-submodule of $\rho$ invariants.

This induces a factorization of $R(x)$,

$$R(x) = (x-1)^{2n_0} Q(x),$$

where $Q(x) = \Pi_{i=1}^{p-1} (x - \xi^i)^{n_i + n_{p-i}}$ must be the characteristic polynomial of $\rho$ acting on $\overline{E}$. But the minimal polynomial of $\rho\,|_{\overline{E}}$ is $x^{p-1} + \cdots + x + 1$. Thus, we conclude

$$Q(x) = (x^{p-1} + \cdots + x + 1)^r,$$

and the proposition follows from this identity. ∎

The next proposition gives a simple necessary condition for inclusions between varieties $A_g(p, \rho_n)$.

PROPOSITION 2.2. *Assume $A_g(p, \rho_n) \subset A_g(q, \sigma_n)$; assume moreover that in the decomposition of V associated to $A_g(p, \rho_n)$, $n_0 = 0$. Then $q \leq p$.*

*Proof.* We use the notation introduced in Subsection 1.1. Let $X_0$ be the generic point of $A_g(p, \rho_n)$ and denote by $\mathfrak{a}_p$ (resp. $\mathfrak{a}_q$) the ideal defining the local deformation algebra of $A_g(p, \rho_n)$ (resp. $A_g(q, \sigma_n)$) in the point $X_0$; both ideals are considered like ideals of $\mathcal{O}$. Then we have an inclusion $\mathfrak{a}_q \subset \mathfrak{a}_p$. We fix the base in $V$ such that $\rho$ in that base is diagonal. Let $A = (a_{ij})$ be the matrix associated to $\sigma$ in this base. Assume that, for example, $t_{12} \notin \mathfrak{a}_p$; this is equivalent to saying that $k_1 = p - k_2$. The inclusion of ideals implies that $t_{12} \notin \mathfrak{a}_q$.

Now, from the relations

$$T \cdot A = \overline{A}^t \cdot T,$$

and the condition $t_{12} \neq 0$ in $\mathcal{O}/\mathfrak{a}_q$ it is easy to deduce that

$$A = \begin{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} & 0 \\ 0 & A' \end{pmatrix}.$$

The relation

$$\begin{pmatrix} t_{11} & t_{12} \\ t_{12} & t_{22} \end{pmatrix} \cdot \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} t_{11} & t_{12} \\ t_{12} & t_{22} \end{pmatrix} \cdot \begin{pmatrix} \bar{a}_{11} & \bar{a}_{12} \\ \bar{a}_{12} & \bar{a}_{22} \end{pmatrix}$$

together with $t_{12} \neq 0$ implies that the eigenvalues associated to

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

are complex conjugated. Thus, to each pair of conjugate eigenvalues of $d\rho$ corresponds a pair of conjugate eigenvalues of $d\sigma$. Using Proposition 2.1 we conclude that $q \leq p$. ∎

Now, we give two examples of this kind of inclusion for $q = 2$. Instead of the fine moduli space constructed in Subsection 1.2 we work on the Siegel half-plane. Note that $\mathbb{H}(p, \rho_n)$ can be identified with the analytical set $Fix(\bar{\rho})$, where $\bar{\rho}$ denotes the symplectic matrix associated to $\rho$.

EXAMPLE 2.3. Let $g = p$ be an odd prime number. Let $V$ be a complex vector space of dimension $g$ and consider the linear map

$$\alpha : V \to V,$$

$$\alpha(e_1) = e_2, \alpha(e_2) = e_3, \dots, \alpha(e_g) = e_1.$$

The decomposition of $d\alpha$ into irreducible representations is

$$V = V_0 \oplus V_1 \oplus \cdots \oplus V_{g-1}.$$

We note that $n_i = \dim V_i = 1$ for all $i \in \{0, 1, 2, \dots, g-1\}$.

As the representation $\alpha$ is integer valued, $\alpha$ and the direct sum with their dual determine a symplectic representation $\theta$. The dimension of $\mathbb{H}_{g,n}(g, \theta)$ is $((g-1)/2) + 1$.

Moreover, it is possible to give a explicit description of the fixed points $\mathbb{H}_{g,n}(g, \theta)$ in the Siegel space.

We will denote by $T$ the nilpotent matrix $T = \{m_i^j\}$ where $m_i^{i+1} = 1$ for $i \in \{1, 2, \dots, g-1\}$ and $m_i^j = 0$ otherwise, and denote $U_i = T^i + T^{g-i}$.

We have

$$\mathbb{H}_{g,n}(g, \theta) = \left\{ Z \in \mathbb{H}_g \mid Z = a_0 I + \sum_{i=1}^{g-1} a_i (U_i + U_i^t) \right\}.$$

On the other hand, the linear map

$$\beta(e_1) = e_g, \beta(e_2) = e_{g-1}, \dots, \beta(e_{g-1}) = e_2, \beta(e_g) = e_1$$

induces a representation:

$$\beta : \mathbb{Z}_2 \hookrightarrow GL(V).$$

The representation is integer valued, and the direct sum of $\beta$ and its dual determine a symplectic representation $\eta$. It can be proved that $\eta$ restricted to $\mathbb{H}_g(g, \theta)$ is the identity. Then we have $\mathbb{H}_g(g, \theta) \subset \mathbb{H}_g(2, \eta)$.

EXAMPLE 2.4. Let $v_1, v_2, \dots, v_g$ be a basis for the irreducible root system $A_g$ and we denote by $\sigma_1, \sigma_2, \dots, \sigma_g$ the associated reflection. Then they are involutions that are all conjugated in the Weyl group $W(A_g)$. We denote $\alpha = \sigma_1\sigma_2\cdots\sigma_g$. $\alpha$ is a Coxeter element, $\alpha^{g+1} = I$ (see [2]). Assume that $g+1$ is a prime number.

The representation of $d\alpha$ in the $g$-dimensional complex vector space $V$ is given by

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_g.$$

Note that $n_i = \dim V_i = 1$ for each $i \in \{1, 2, 3, \dots, g\}$.

As the representation $\alpha$ is integer valued, $\alpha$ and the direct sum with their dual determine a symplectic representation $\theta$ and $\mathbb{H}_g(g+1, \theta)$ is a smooth connected subvariety of dimension $g/2$.

As in Example 2.3 it is possible to give a description of the fixed points in Siegel space. Assume $g = 2k$.

We have

$$\mathbb{H}_{g,n}(g+1, \theta) = \left\{ Z \in \mathbb{H}_g \mid Z = -k\left(\sum_{i=1}^{i=k} a_i\right)I + \sum_{i=1}^{i=k} a_i(U_i + U_i^t) \right\}.$$

The linear map

$$\beta(e_1) = e_g, \beta(e_2) = e_{g-1}, \dots, \beta(e_{g-1}) = e_2, \beta(e_g) = e_1$$

and the direct sum with their dual induce a representation:

$$\eta : \mathbb{Z}_2 \to Sp(2g, \mathbb{Z}).$$

It can be proved that $\eta$ is in the normalizer $N_{Sp}(\theta)$ and it can induce the identity in $\mathbb{H}_g(g, \theta)$. Then we have $\mathbb{H}_g(g+1, \theta) \subset \mathbb{H}_g(2, \eta)$.

We finish this section by computing the number of irreducible components of the form $A_{g,n}(2, \rho)$. This number will be denoted by $\# A_{g,n}(2, \rho)$.

PROPOSITION 2.5. *Let $(X, \Theta)$ be a principally polarized abelian variety of dimension $g$. Then we have $\# A_{g,n}(2, \rho) = ((g+2)/2)^2$ if $g$ is even and $\# A_{g,n}(2, \rho) = (g+1)(g+3)/4$ if $g$ is odd.*

*Proof.* The matrices

$$A(a, b, c) = U(a, b, c) \oplus U(a, b, c)^t$$

give a complete set of nonconjugated involutions in $Sp(2g, \mathbb{Z})$, where

$$U(a, b, c) = \oplus^a W \oplus (-I)_b \oplus I_c,$$

$I_c$ is the $c \times c$-identity, $(-I)_b$ is the $b \times b$-minus identity, $W$ is the matrix

$$W = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix},$$

and $a$, $b$, and $c$ are non-negative integer solutions of

$$2a + b + c = g.$$

A computation on the generating function

$$L(t) = \frac{1}{(1-t^2)(1-t)^2}$$

gives that the number of solutions is

$$N = \frac{1}{2}\left[\frac{(g+1)(g+3)}{2} + \frac{1}{4}(1 + (-1^g)\right]$$

and we obtain the result. ∎

2.2. *Determination of Some Irreducible Components.* Note that $\rho_n$ determines uniquely the action of $\rho$ on the lattice $U \subset V$ defining $X$ as a complex torus. Let $E$ be the integral representation of $\rho$ in $U$.

Let $H = \langle \rho \rangle$ be the group of automorphisms generated by $\rho$ and $\mathbb{Z}[H]$ its group ring over $\mathbb{Z}$:

$$\mathbb{Z}[H] \simeq \mathbb{Z}[x]/(x-1)(x^{p-1} + \cdots + x + 1).$$

Let us denote $A = \mathbb{Z}[x]/(x^{p-1} + \cdots + x + 1) = \mathbb{Z}[\xi]$.

We will assume in the rest of the paper that $n_0 = 0$. Under this assumption the integral representation $E$ of $\rho$ satisfies that $1 + \rho + \rho^2 + \cdots + \rho^{p-1} \in \operatorname{Ann}(E)$; moreover $E$ is a regular $A$-module and verifies:

PROPOSITION 2.6. *The $A$-module $E$ is isomorphic to $A \oplus \cdots \oplus A \oplus I$, for some fractional ideal $I$ of $A$.*

*Proof.* It follows from [9].

Let us recall some results proved in [1] concerning the automorphism group in the critical case $g = (p-1)/2$.

In this case, all p.p.a.v., $X = V/U$ of dimension $g$ with automorphism of order $p = 2g+1$ can be explicitly constructed.

The decomposition of $V = T_0(X)$ as a $H$-module is

$$V = V_{k_1} \oplus \cdots \oplus V_{k_g},$$

where $\dim V_{k_1} = \cdots = \dim V_{k_g} = 1$.

This decomposition is characterized by the set $C = \{k_1, \ldots, k_g\}$ which can be considered like a subset of $\mathbb{F}_p^*$. The allowed sets $C \subset \mathbb{F}_p^*$ defining all the possible representations of $H$ in $V$ are the so-called CM-type (here we use the same notation as in [1]):

$C \subset \mathbb{F}_p^*$ such that

$$C \cap (-C) = \emptyset, \qquad C \cup (-C) = \mathbb{F}_p^*. \tag{2.1}$$

The immersion of $\mathbb{Q}(\xi)$ in $\mathbb{C}$ defined by $\phi_{k_1}, \ldots, \phi_{k_g}$ $(\phi_k(\xi) := \xi^k)$ induces an isomorphism of $\mathbb{R}$-vector spaces,

$$\mathbb{Q}(\xi) \otimes_{\mathbb{Q}} \mathbb{R} \xrightarrow{\simeq} \mathbb{C}^g. \tag{2.2}$$

The lattice $U$ defining $X$ is a fractional ideal $I_U$ of $\mathbb{Q}(\xi)$ and $X$ can be reconstructed as the complex torus:

$$(\mathbb{Q}(\xi) \otimes_{\mathbb{Q}} \mathbb{R})/I_U.$$

Let us observe that $A_{(p-1)/2,\, n}(p, \rho_n)$ is a finite set of points of $A_{(p-1)/2,\, n}$; this can be deduced, for example, from the local deformation theory developed in the previous section.

The automorphism groups of the points of $A_{(p-1)/2,\, n}(p, \rho_n)$ are characterized in [1, Theorem 2].

Let us fix a subset $C = \{k_1, \ldots, k_g\} \subset \mathbb{F}_p^*(g = (p-1)/2)$, verifying the above conditions. A fractional ideal $I$ of $\mathbb{Q}(\xi)$ defines a lattice $U_I$ and assume that the resulting abelian variety is principally polarized (it can be ensured with some extra unimodularity hypothesis on $I$, see [1] for more details). If $G$ is the automorphism group of $(X, \Theta)$ and $G_+ = G/\{\pm 1\}$ there are two possibilities for the group $G_+$:

(a) $G_+$ is a semidirect product of $\mathbb{Z}_p$ and $\mathbb{Z}_t$, with $t$ an odd divisor of $(p-1)/2$; this case can be characterized like the case in which $G_+$ is solvable and has a unique $p$-Sylow subgroup.

(b) $G_+ \simeq PSL_2(\mathbb{F}_p)$; in this case $p \equiv -1 \bmod(4)$.

Returning to the general case, let $V = V_1 \oplus \cdots \oplus V_{p-1}$ be the decomposition corresponding to the moduli space $A_{g,\, n}(p, \rho_n)$ (recall that we assume $n_0 = 0$, or equivalently $V_0 = 0$),

$$V = (V_1 \oplus V_{p-1}) \oplus \cdots \oplus (V_{(p-1)/2} \oplus V_{(p+1)/2}),$$

$\dim(V_i \oplus V_{p-i}) = r$, $g = r(p-1)/2$, $g' = (p-1)/2$.

By selecting a vector space of dimension 1 in each component $V_i \oplus V_{p-i}$ one can decompose $V$ as

$$V = W_1 \oplus \cdots \oplus W_r,$$

where $W_i = W_{i,\, k_1^i} \oplus \cdots \oplus W_{i,\, k_{g'}^i}$ $(\dim W_{i,\, k_j^i} = 1)$ is a decomposition defined by a subset

$$C_i = \{k_1^i, \ldots, k_{g'}^i\} \subset \mathbb{F}_p^*.$$

The decomposition $V = W_1 \oplus \cdots \oplus W_r$ is not unique in general. Let us denote

$$W^h = W_1^h \oplus \cdots \oplus W_r^h \qquad (h = 1, \ldots, M)$$

the set of all different decompositions of $V$ and $\{C_1^h, \ldots, C_r^h\}$ the subsets of $\mathbb{F}_p^*$ associated to the decomposition $W^h$.

LEMMA 2.7. *Given the decompositions $V = V_1 \oplus \cdots \oplus V_{p-1}$ and $E = A \oplus \cdots \oplus A \oplus I$ determined by the representation $\rho_n$, the moduli space $A_{g,n}(p, \rho_n)$ contains the points*

$$Z_h = (X_1^h \times \cdots \times X_r^h, \Theta_h),$$

*where $(X_i^h, \Theta_i^h)$ is a p.p.a.v. of dimension $(p-1)/2$ admitting a cyclic automorphism of order $p$ with CM-type $C_i^h$.*

*Proof.* Given the integer representation $E$ of $\rho$ consider the decomposition

$$E = A \oplus \cdots \oplus A \oplus I,$$

and the associated diagonal representation of $d\rho = diag(\xi_1^k, \ldots, \xi_g^k)$. Decompose the set $\{k_1, \ldots, k_g\}$ into $r$ sets satisfying conditions 2.1 (it is possible according to Proposition 2.1).

Thus the problem reduces to constructing a p.p.a.v of dimension $g' = (p-1)/2$ admitting an automorphism $\rho$ of order $p$ with fixed CM-type and integer representation. This can be done using the arguments in [1, pp. 410–411; 5, Sect. 22]: the ideal $I$ is a free $\mathbb{Z}$-module of rank $2g'$. Use the maps $\phi_{k_i}$ in order to obtain an isomorphism as in Subsection 2.2. Then define a bilinear skew-symmetric form with integer values over the image $U_I$ of $I$. ∎

LEMMA 2.8. *Let $p$ be an odd prime number and $g = (p-1)/2$. All the p.p.a.v. defining the moduli spaces $A_{g,n}(p, \rho_n)$ are irreducible polarized abelian varieties.*

*Proof.* Let $(X, \Theta)$ be a p.p.a.v. of dimension $g = (p-1)/2$ such that there exists an automorphism $\rho \in Aut(X, \Theta)$ of order $p$, and let $H = \langle \rho \rangle$ be the subgroup of $Aut(X, \Theta)$ generated by $\rho$. From the constructions in [1] and the results of [9], $X = V/E$ where the lattice $E$ is an irreducible $\mathbb{Z}[H]$-module. Let us assume that

$$(X, \Theta) = (X_1, \Theta_1) \times \cdots \times (X_r, \Theta_r),$$
$$\Theta = \pi_1^* \Theta_1 + \cdots + \pi_r^* \Theta_r,$$

is a decomposition of $(X, \Theta)$ as a product of irreducible p.p.a.v. This decomposition would induce a decomposition of the lattice $E$,

$$E = E_1 \oplus \cdots \oplus E_r, \qquad rkE = 2g = p - 1.$$

From the irreducibility of $E$ as a $\mathbb{Z}[H]$-module it follows that $H$ acts transitively on the set $\{E_1, \ldots, E_r\}$. But this implies that $r = 1$, since $rkE = p - 1$ and $|H| = p$. ∎

*Remark* 2.9. Lemma 2.8 generalizes to all moduli spaces $A_{(p-1)/2, n} \times (p, \rho_n)$ the result proved in [1, Proposition 2] about the irreducibility of p.p.a.v. of dimension $(p-1)/2$ whose automorphism group is $PSL_2(\mathbb{F}_p) \times \{\pm 1\}$.

LEMMA 2.10. *For a general point $(X, \Theta) \in A_{g,n}(p, \rho_n)$, $Aut(X, \Theta)$ is a subgroup of $Aut(Z_h, \Theta_h)$.*

*Proof.* Let $\mathscr{V}$ be a discrete valuation ring and $S = \operatorname{Spec} \mathscr{V}$. Let us consider a $S$-valued point of $A_{g,n}(p, \rho_n)$.

$$f : S \to A_{g,n}(p, \rho_n),$$

$S = \{\eta_0, \eta_1\}$, $\eta_0 =$ closed point, $\eta_1 =$ generic point.

$f(\eta_1) =$ generic point of $A_{g,n}(p, \rho_n) = (X, \Theta)$, $f(\eta_0) = (Z_h, \Theta_h)$.

The morphism $f$ defines an abelian scheme (see Proposition 1.3)

$$\mathscr{X} \xrightarrow{p} S$$

such that $p^{-1}(\eta_0) = Z_h$ and $p^{-1}(\eta_1) = X$. Then, one can define a homomorphism of groups

$$\Phi : Aut(X, \Theta) \to Aut(Z_h, \Theta_h)$$

as follows: given $\rho \in Aut(X, \Theta)$, let $\bar{\rho}$ be the extension of $\rho$ to the abelian scheme $\mathscr{X}$, and $\Phi(\rho)$ be the restriction of $\bar{\rho}$ to the closed fibre $p^{-1}(\eta_0)$. Lemma 1.6 implies that $\Phi$ is injective. ∎

*Remark* 2.11. Let us consider a point $(Z_h, \Theta_h) \in A_{g,n}(p, \rho_n)$ as defined in Lemma 2.7, and let $Z_h = X_1 \times \cdots \times X_r$ be the decomposition of $Z_h$; from Lemma 2.8 the p.p.a.v. $(X_i, \Theta_i)$ are irreducible abelian varieties. As some varieties $X_i$ of the decomposition could appear several times one can rewrite the decomposition of $Z_h$ as

$$Z_h = X_1^{s_1} \times \cdots \times X_t^{s_t},$$

where $X_i \ncong X_j$ for $i \neq j$, and $s_1 + \cdots + s_t = r$. Note that

$$Aut(Z_h, \Theta_h) = [S_{s_1} \rtimes Aut(X_1, \Theta_1)^{s_1}] \times \cdots \times [S_{s_t} \rtimes Aut(X_t, \Theta_t)^{s_t}],$$

where $S_j$ is the $j$-symmetric group. Denote this group by $G(C_h)$. From Lemma 2.10 we know that given a general point $(X, \Theta) \in A_{g,n}(p, \rho_n)$

$$Aut(X, \Theta) \subset G(C_h) = [S_{s_1} \rtimes Aut(X_1, \Theta_1)^{s_1}] \\ \times \cdots \times [S_{s_t} \rtimes Aut(X_t, \Theta_t)^{s_t}] \tag{2.3}$$

for every $Z_h = X_{h_1}^{s_1} \times \cdots \times X_{h_t}^{s_t} \in A_{g,n}(p, \rho_n)$.

This fact reduces the possible automorphism groups of the general points of the components $A_{g,n}(p, \rho_n)$.

Given the decomposition $V = V_1 \oplus \cdots \oplus V_{p-1}$ ($n_i = \dim V_i$) associated with $A_{g,n}(p, \rho_n)$ let us consider all the possible subsets $C_h = \{C_1^h, \ldots, C_r^h\}$ ($h = 1, \ldots, M$) of $\mathbb{F}_p^* \times \cdots \times \mathbb{F}_p^*$ which define decompositions

$$V = W_1^h \oplus \cdots \oplus W_r^h \tag{2.4}$$

as described above.

Following the notations of [1], given a subset $C \subset \mathbb{F}_p^*$ defining $A_{(p-1)/2,n}(p, \rho_n)$, let us define

$$\Delta(C) = \{k \in \mathbb{F}_p^* : k \cdot C = C\}, \tag{2.5}$$

$-1 \notin C$, and $|\Delta(C)|$ is an odd number.

Let us consider a point $(X, \Theta, \rho) \in A_{g,n}(p, \rho_n)$, $H = \langle\rho\rangle$ the group of automorphisms of $(X, \Theta)$ of order $p$ generated by $\rho$, and $G = Aut(X, \Theta)$ the full automorphism group of $(X, \Theta)$. Let $N_G(H)$ be the normalizer of $H$ in $G$; one can define an exact sequence of groups,

$$1 \to Z_G(H) \to N_G(H) \xrightarrow{\Phi_X} \mathbb{F}_p^*, \tag{2.6}$$

where $Z_G(H)$ is the centralizer of $H$ and given $g \in N_G(H)$, $g\rho g^{-1} = \rho^k$, $\Phi_X(g) :=$ class of $k$ in $\mathbb{F}_p^*$.

LEMMA 2.12. *Let $(X, \Theta, \rho)$ be a general point of $A_{g,n}(p, \rho_n)$, then the inclusions $G \hookrightarrow G(C_h)$ (for every $h = 1, \ldots, M$) induce inclusions*

$$H \hookrightarrow S_{s_i^h} \rtimes Aut(X_{h_i}, \Theta_{h_i})^{s_i^h}$$

*for every $h = 1, \ldots, M$ and $i = 1, \ldots, t$.*

*Moreover, one has an inclusion*

$$\operatorname{Im} \Phi_X \subset \bigcap_{i,h} \Delta(C_i^h) =: \Delta(g, p, \rho) \subset \mathbb{F}_p^*.$$

*Proof.* The inclusions $\langle\rho\rangle \hookrightarrow S_{s_i^h} \rtimes Aut(X_{h_i}, \Theta_{h_i})^{s_i^h}$ are the composition of the natural immersion $\langle\rho\rangle \hookrightarrow G(C_h)$ with the natural projections on $S_{h_i} \rtimes Aut(X_{h_i}, \Theta_{h_i})^{s_i^h}$; these morphisms are injective, since $n_0 = 0$.

The another inclusion, $Im\Phi_X \subset \Delta(g, p, \rho)$ is a direct consequence of the above fact and the definitions of $\Phi_X$ and $\Delta(C)$. ∎

PROPOSITION 2.13. *Let* $(X, \Theta, \rho)$ *be a general point of* $A_{g,n}(p, \rho_n)$, $V = V_1 \oplus \cdots \oplus V_{p-1}$ *the associated decomposition of* $V = T_0(X)$, $n_i = \dim V_i$, *and* $r = n_i + n_{p-i}$. *If either* $r \geq 3$ *and* $\dim A_{g,n}(p, \rho_n) \geq 1$, *or* $r = 2$ *and* $\dim A_{g,n}(p, \rho_n) = 1$, *then* $\Delta(g, p, \rho) = \{1\}$ *and* $N_G(H) = Z_G(H)$.

*Proof.*

$$V = (V_1 \oplus V_{p-1}) \oplus \cdots \oplus (V_{(p-1)/2} \oplus V_{(p+1)/2}) \simeq W_1 \oplus \cdots \oplus W_r,$$
$$r = \dim V_i + \dim V_{p-i},$$

if $r \geq 3$ and $\dim A_{g,n}(p, \rho_n) \geq 1$ or $r = 2$ and $\dim A_{g,n}(p, \rho) = 1$ and we can define a decomposition $V \simeq W_1 \oplus \cdots \oplus W_r$, such that

$$C(W_1) = \{k_1, \ldots, k_{g'}\}, \qquad g' = g/r,$$
$$C(W_2) = \{-k_1, k_2, \ldots, k_{g'}\}.$$

Obviously $\Delta(k_1, \ldots, k_{g'}) \cap \Delta(-k_1, k_2, \ldots, k_{g'}) = \{1\}$. The result follows from Lemma 2.12. ∎

Let us define now another numerical invariant associated to the moduli space $A_{g,n}(p, \rho_n)$.

Given $V = V_1 \oplus \cdots \oplus V_{p-1}$ ($n_i = \dim V_i, n_i + n_{p-i} = r$) determined by $A_{g,n}(p, \rho_n)$ let us define the following subset of $\mathbb{F}_p^* \times \overset{g'}{\cdots} \times \mathbb{F}_p^*$,

$$\mathrm{Spec}(V) := \{k \in \mathbb{F}_p^* : \dim V_k \neq 0\}.$$

$\overline{C}(g, p, n_1, \ldots, n_{p-1}) \subset \mathbb{F}_p^* \times \overset{g'}{\cdots} \times \mathbb{F}_p^*$ is defined by $C = \{k_1, \ldots, k_{g'}\} \in \overline{C}(g, p, n_1, \ldots, n_{p-1})$ if and only if it satisfies the following conditions:

(1) $C \cap -C = \emptyset, \quad C \cup -C = \mathbb{F}_p^*$,

(2) $C \subset \mathrm{Spec}(V)$.

Given a subset $A \subset \overline{C}(g, p, n_1, \ldots, n_{p-1})$ let us define

$$n_i(A) = |\{C \in A : i \in C\}|, \qquad i \in \{1, \ldots, p-1\}.$$

Now we define

$$C(g, p, n_1, \ldots, n_{p-1}) = \{A \subset \overline{C}(g, p, n_1, \ldots, n_{p-1}) \mid n_i(A) \leq n_i, \forall i \in \{1, \ldots, p-1\}\},$$

and the number

$$N(g, p, n_1, \ldots, n_{p-1}) = Sup\{|A|; A \in C(g, p, n_1, \ldots, n_{p-1})\}.$$

In other words, $N(g, p, n_1, \ldots, n_{p-1})$ is the maximal number of different CM-types in which we can decompose $\mathrm{Spec}(V)$. Let us observe that $N(g, p, n_1, \ldots, n_{p-1}) \leq r$.

THEOREM 2.14. *Let us consider the decomposition* $V = V_1 \oplus \cdots \oplus V_{p-1}$ $(n_i = \dim V_i, r = n_i + n_{p-i})$ *determined by the moduli space* $A_{g,n}(p, \rho_n)$ *and let* $(X, \Theta, \rho)$ *be a general point of* $A_{g,n}(p, \rho_n)$.

*If we assume that* $N(g, p, n_1, \ldots, n_{p-1}) = r$, *one has the following possibilities for the group* $G_+(X) = Aut(X, \Theta)/\{\pm 1\}$.

(1) $G_+(X) = \mathbb{Z}_p$ *if* $r \geq 3$ *and* $\dim A_{g,n}(p, \rho) \geq 1$ *or* $r = 2$ *and* $\dim A_{g,n}(p, \rho) = 1$.

(2) *If* $r = 2$ *and* $\dim A_{g,n}(p, \rho) = \sum_i n_i n_{p-i} > 1$ *we have two possibilities*:

(2a) $G_+$ *is the semidirect product of* $H = \langle \rho \rangle$ *and a cyclic subgroup of* $\Delta(g, p, \rho)$ *of order* $t$, *where* $t$ *is an odd a divisor of* $(p-1)/2$.

(2b) $G_+ \simeq PSL_2(\mathbb{F}_p^*)$. *In this case* $p \equiv -1 (\mathrm{mod} 4)$.

*Proof.* The condition $N(g, p, n_1, \ldots, n_r) = r$ means that we can find $(C_1, \ldots, C_r) \in \overline{C}(g, p, n_1, \ldots, n_r)$ such that the p.p.a.v. $(\mathbb{Z}, \Theta_Z)$ defined by this sequence admits a decomposition,

$$Z = X_1 \times \cdots \times X_r$$

with $(X_i, \Theta_i) \ncong (X_j, \Theta_j)$ for $i \neq j$ and then

$$Aut(Z, \Theta_Z) = Aut(X_1, \Theta_1) \times \cdots \times Aut(X_r, \Theta_r).$$

From Lemma 2.10 and the condition $n_0 = 0$, we have that

$$Aut(X, \Theta) \subset Aut(X_i, \Theta_i)$$

for all $i$.

Now, we can apply the classification theorem for $G_i = Aut(X_i, \Theta_i)$. Note that $Z_{G_{i_+}}(H) = H$; thus $Z_{G_+}(H) = H$. From Proposition 2.13 we conclude that $N_G(H) = H$, and it follows from the results in [1] that $G_+ = H$.

(2) This case can be easily deduced from Theorem 2 of [1] and Proposition 2.13. ∎

We have obtained:

COROLLARY 2.15. *Under the hypothesis of Theorem* 2.14, *if* $r \geq 3$ *and* $\dim A_{g,n}(p, \rho) \geq 1$ *or* $r = 2$ *and* $\dim A_{g,n}(p, \rho) = 1$ *then* $A_g(p, \rho_n)$ *is an irreducible component of* Sing $A_g$.

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